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ON EXISTENCE AND UNIQUENESS FOR A NEW CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING COMPACTNESS METHODS AND DIFFERENTIAL DIFFERENCE SCHEMES(1)

BY

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ABSTRACT. We prove existence and uniqueness results for the following Cauchy problem in the half plane $t \geq 0$: $u_t + (f(u))_x + u_{xxx} = g_1(u)u_{xx} + g_2(u)(u_x)^2 + p(u)$, $u(x, 0) = u_0(x)$, where $u = u(x, t)$ and the subscripts indicate partial derivatives. We require that f, g_1, g_2, and p be sufficiently smooth and satisfy $f'(u) \geq 0, \int_0^u f(v)\,dv \geq 0$, and other similar sign conditions on g_1, g_2, and p. Our hypotheses allow for exponential growth of f, g_1, g_2, and p so long as the sign conditions are satisfied and include the special cases $f(u) = u^{2n+1}$, $g_1(u) = u^{2m}$, $g_2(u) = -u^{2r+1}$, and $p(u) = -u^{2s+1}$, for n, m, r, and s non-negative integers.

To obtain a global solution in time, we perturb the equation by $-\epsilon(u_{xxxx} - (f(u))_{xx})$. The perturbed equation is solved locally (in time) and this solution is extended to a global solution by means of a priori estimates on the H^s (of space) norms of the local solution. These estimates require the use of new nonlinear functionals. We then obtain the solution to the original equation as a limit of solutions to the perturbed equation as ϵ tends to zero using the standard techniques.

For the related periodic problem, for which we require $u(x + 2\pi, t) = u(x, t)$ for all $t \geq 0$, we also obtain existence and uniqueness results. We prove existence for this problem via similar techniques to the nonperiodic case.

We then consider differential difference schemes for the periodic initial value problem and show that we may obtain the solution as the limit of solutions to an appropriate scheme.

1. Introduction and preliminaries. The generalized Korteweg-de Vries (K-dV) equation we shall consider is

$$u_t - (f(u))_x + u_{xxx} = g_1(u)u_{xx} + g_2(u)(u_x)^2 + p(u) \tag{1.1}$$

where $u = u(x, t)$ and the subscripts indicate partial derivatives. In this paper

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we shall deomonstrate existence and uniqueness results for global solutions of the initial value problem for (1.1) in $t \geq 0$ taking data

$$u(x, 0) = u_0(x). \tag{1.2}$$

We further discuss the closely related periodic initial value problem (1.1), (1.2) and

$$u(x, t) = u(x + 2\pi, t) \quad \text{for } t \geq 0. \tag{1.3}$$

These results are the first of their kind for nonlinear functions g_1 and g_2.

The problems (1.1), (1.2) and (1.1), (1.2), (1.3) have been studied most recently by Tsutsumi, Mukasa, and Iino [11], for functions f satisfying $f' \geq 0$, $\int_0^u f(v)\,dv \geq 0$ and for $g_1 = 1$, $g_2 = p = 0$. Previous to this, Sjöberg [9] and Temam [10] have studied the K-dV itself $(f(u) = u^2/2,\ g_1 = q_2 = p = 0)$ by different methods. Kruskal, Miura, Gardner, and Zabusky [6], [12] have done fundamental work in the discovery of conservation laws for the K-dV and some of its generalizations and have done other work in the description of solutions to the K-dV. Lax [5] has investigated the long term interaction of progressive waves for the K-dV.

The difficult part of our development, as in all previous work on the K-dV and its generalizations, is in obtaining a priori estimates for the norms of solutions to the problem. As an important example, the conservation laws of Kruskal, Miura, *et al.* make available a priori estimates for the K-dV and their generalizations of it.

In §4, we will obtain the solution to the problem (1.1), (1.2) as a limit of solutions to the perturbed problem

$$u_t - (f(u))_x + u_{xxx} = g_1(u)u_{xx} + g_2(u)(u_x)^2 + p(u) - \epsilon(u_{xxxx} - f(u)_{xx}) \tag{1.4}$$

with data (1.2). Because of the nonlinearity in the second order terms, we may use the perturbation $-\epsilon u_{xxxx}$ to obtain a local solution to a perturbed problem. However, in order to extend the local solution to a global solution using a priori estimates, we find the perturbation $-\epsilon(u_{xxxx} - f(u)_{xx})$ requires much less of the function f than the perturbation $-\epsilon u_{xxxx}$.

In the last section, we show the uniqueness of a solution to (1.4), (1.2) or (1.1), (1.2). There we will also state theorems for the periodic initial-value problem (1.1), (1.2), (1.3) analogous to the initial-value problem (1.1), (1.2).

Remark. Temam [10] was the first to use the fourth order perturbation $-\epsilon u_{xxxx}$ to solve the K-dV itself, though his techniques were somewhat different from ours.

Definitions and useful lemmas. For $f \in L^2(R^1)$, let $\|f\|^2 = \int f^2(x)\,dx$. If $f(x)$ is measurable and bounded, let $\|f\|_\infty = \text{ess sup}|f(x)|$ $(-\infty < x < \infty)$. For s a nonnegative integer, let H^s be the space of functions u in $L^2(R^1) = L^2$ having weak L^2-derivatives $D^k u$ of orders $k = 1, \cdots, s$ having norm $\|u\|_s^2 = \Sigma_{k\leq s}\|D^k u\|^2$. The space H^s has an equivalent norm, $\|u\|_{H^s} = \|(1+\xi^2)^{s/2}\hat{u}(\xi)\|$, where $\hat{u}$ is the Fourier transform of u, and we will use the fact that this generalizes to spaces H^s, for s any real number. Let C^k be those functions k times continuously differentiable in R^1 and let $C^\infty = \bigcap_{k=1}^\infty C^k$. We will say $u(x, t) \in L^\infty(0, T; H^s)$ if u as a function of x is in H^s for each t, $0 \leq t \leq T$, and $\sup\|u(\cdot, t)\|_s < \infty$ $(0 \leq t \leq T)$.

For $f(u)$ locally integrable, let $If(u) = \int_0^u f(v)\,dv$. Let $f(u)_x$ denote $(f(u))_x$, u_x^2 denote $(u_x)^2$, u_{xx}^2 denote $(u_{xx})^2$, etc.

Lemma 1.1. *If $u(x) \in H^s$, $s \geq 1$, then for $p \leq s-1$, $\alpha = (2s)^{-1}(2p+1)$,*

$$\|D^p u\|_\infty \leq C_1(\|u\|^{1-\alpha}\|D^s u\|^\alpha + \|u\|) \leq C_2\|u\|_s \tag{1.5}$$

where C_1, C_2 are independent of u.

Lemma 1.2. *Let $k \geq 1$ and let $f(u) \in C^k$, $f(0) = 0$, $u(x, t) \in L^\infty(0, T; H^k)$, then $f(u(x, t)) \in L^\infty(0, T; H^k)$ and we have*

$$\|f(u(t))\|_1 \leq cM_1(f, b)\|u(t)\|_1, \tag{1.6}$$

$$\|f(u(t))\|_k \leq cM_k(f, b)(1 + \|u(t)\|_{k-1}^{k-1})\|u(t)\|_k, \tag{1.7}$$

where $M_k(t, b) = \max_s \sup_v |D^s f(v)|$ $(s = 1, \cdots, k;\ |v| \leq b)$ for $b = \sup_\tau \|u(\tau)\|_\infty$ $(0 \leq \tau \leq t)$. The constants appearing are independent of f and u.

Remarks on the proof of Lemma 1.1. We prove (1.5) first for functions v having compact support using a Sobolev type result following Agmon [1, p. 32 and p. 209]. Fix x_0; using an appropriate test function ϕ of compact support so that $\phi \equiv 1$ is a neighborhood of the origin, we apply the result to $v = \phi(x - x_0)u$ from which follows the lemma.

2. Local existence theorem for the perturbed problem. To solve the problem (1.4), (1.2) we linearize (1.4) and obtain

Theorem 2.1. *Let $u_0 \in H^{s-1}$ for $s \geq 3$ or $u_0 \in H^s$ for $s = 2$ and $f \in C^{s-1}$, p, g_1, $g_2 \in C^{s-2}$, $f(0) = 0$. The problem (1.4), (1.2) has a solution $u(x, t) \in L^\infty(0, T; H^s)$ for some $T = T^s$.*

Proof. Let ϕ be a solution to the (linear) problem

$$\phi_t + \phi_{xxx} = -\epsilon\phi_{xxxx}, \quad \phi(x, 0) = u_0(x). \tag{2.1}$$

For $n = 1, 2, \cdots$, let the sequence $\{u^n\}$ be defined by the (linearized) problem

$$u_t^n - f(v^{n-1})_x + u_{xxx}^n = g_1(v^{n-1})v_{xx}^{n-1} + g_2(v^{n-1})(v_x^{n-1})^2$$

$$(2.2) \qquad + p(v^{n-1}) - \epsilon(u_{xxxx}^n + f(v^{n-1})_{xx}),$$

$$u^n(x, 0) = 0$$

where $u^0 \equiv 0$ and $v^{n-1} = \phi + u^{n-1}$. Notice by adding (2.1) and (2.2) that if $\lim_n u^n$ can be taken in some appropriate H^s sense, then it will be at least a distribution solution of (1.4), (1.2). To prove this is in fact the case we are led from (2.2) to the problem

$$(2.3) \qquad w_t + w_{xxx} = -\epsilon w_{xxxx} + a(x, t), \qquad w(x, 0) = 0.$$

For $w = u^n$ and $a(x, t) = a_{n-1}(x, t)$ where

$$a_{n-1}(x, t) = f(v^{n-1})_x + g_1(v^{n-1})v_{xx}^{n-1} + g_2(v^{n-1})(v_x^{n-1})^2$$

$$(2.4) \qquad + p(v^{n-1}) - \epsilon f(v^{n-1})_{xx}$$

$$= f(v^{n-1})_x + b_1(v^{n-1})v_{xx}^{n-1} + b_2(v^{n-1})_{xx}$$

where $b_1(v) = g_1(v) - Ig_2(v)$ and $b_2(v) = IIg_2(v) + IIp(v) + \epsilon f(v)$.

We begin with a lemma about (2.1).

Lemma 2.1. *If $u_0 \in H^s$, then $\phi(x, t) \in L^\infty(0, T; H^s)$ for all $T > 0$ and*

$$(2.5) \qquad \|\phi(t)\|_s \le \|u_0\|_s \quad \textit{for all } t \ge 0.$$

Proof. Let $E(x, t) = \int \exp(-i\xi x - \epsilon\xi^4 t + i\xi^3 t)\,d\xi$ be a fundamental solution to (2.1). Then $\phi(x, t) = E(x, t) * u_0(x)$, where $*$ denotes convolution in x, is a distribution solution to (2.1). To show (2.5), we have $\|D^k\phi(t)\| = \|(i\xi)^k \hat{u}_0(\xi)\hat{E}(\xi, t)\| \le \|D^k u_0\|$. ∎

For problem (2.3), we have

Lemma 2.2. *If $a(x, t) \in L^\infty(0, T; H^{s-3})$, then $w(x, t) \in L^\infty(0, T; H^s)$ and satisfies*

$$(2.6) \qquad \|w(t)\|_k \le C(\epsilon)A(t)\sup_\tau \|a(\tau)\|_{k-3} \ (0 \le \tau \le t), \quad \textit{for } c(\epsilon) > 0$$

for $k = 3, 4, \cdots, s$ where $A(t) = t^{1/4} + t^{1/2} + t^{3/4} + t$. For $k = 0, 1, 2, 3$, the same lemma is true where we replace $\|a(\tau)\|_{k-3}$ by $\|a(\tau)\|_{H^{k-3}}$.

Proof. In the sense of distributions, $w(x, t) = \int_0^t E(x, t-\tau) * a(x, \tau)\,d\tau$ is a solution to (2.3). Using Fourier transforms and estimating, we obtain

$$\|D^k w(t)\| \le \int_0^t \|\exp(-\epsilon\xi^4(t-\tau))(i\xi)^k \hat{a}(\xi, \tau)\|\,d\tau$$

$$\le \int_0^t \|\exp(-\epsilon\xi^4(t-\tau))(\xi^2+1)^{3/2}(\xi^k/(\xi^2+1)^{k/2})(\xi^2+1)^{(k-3)/2}\hat{a}(\xi, \tau)\|\,d\tau.$$

Then using the fact that $(1+\xi^2)^{3/2} \le 1 + |\xi| + \xi^2 + |\xi|^3$, we get

$$\|D^k w(t)\| \le \int_0^t \|\exp(-\epsilon\xi^4(t-r))(1+|\xi|+\xi^2+|\xi|^3)(\xi^2+1)^{(k-3)/2}\hat{a}(\xi,r)\|\,dr$$
$$\le \sum_{i=0}^{3} \int_0^t (\epsilon(t-r))^{-1/4}\,dr \sup_{0\le r\le t} \|z^i \exp(-z^4)(\xi^2+1)^{(k-3)/2}\hat{a}(\xi,r)\|$$

where $z = |\xi|(\epsilon(t-r))^{1/4}$. Since $z^i \exp(-z^4) \le c$ for $i = 0, \cdots, 3$, we obtain (2.6).

Using Lemmas 2.1 and 2.2, we prove the following

Lemma 2.3. *Let $u_0 \in H^2$. For the sequence $\{u^n\}$ defined by the scheme* (2.2) *we have $\|u^n(t)\|_2 \le c(\epsilon)$ for $0 \le t \le t_0$ for all n and some $t_0 > 0$. Let $u_0 \in H^{s-1}$ for $s \ge 3$. Then $\|u^n(t)\|_k \le c(\epsilon, k, t)$ for $0 \le t \le t_0$ for $k = 3, 4, \cdots, s$.*

Proof. We first prove the lemma for $k = 2$. Fix $T_0 > 0$. Using (2.6) for $k = 2$, $w = u^n$, $a(x, t) = a_{n-1}$ is defined in (2.4), and then using the triangle inequality, we obtain

$$\|u^n(t)\|_2 \le cA(t) \sup_{0\le r\le T_0} \{\|f(v^{n-1})\| + \|b_2(v^{n-1})\|_1 + \|b_1(v^{n-1})v^{n-1}_{xx}\|\} \tag{2.7}$$

for $0 \le t \le T_0$. Note that we use the notation $\sup_{0\le r\le T_0}\|b(u)\|_s$ to mean $\sup_{0\le r\le T_0}\|b(u(\cdot, r))\|_s$. We first substitute $n = 1$ in (2.7) so $v^{n-1} = \phi$. By Lemma 1.2, we may estimate $\|f(v^{n-1})\| + \|b_2(v^{n-1})\|_1$ by $c(f, b_2)\|u_0\|_1$. Since $\|v^{n-1}\|_\infty \le c\|v^{n-1}\|_1 \le c\|\phi\|_1 \le c\|u_0\|_1$, and since b_1 is continuous, $\|b(v^{n-1})\|_\infty < c$. Therefore, $\|b_1(v^{n-1})v^{n-1}_{xx}\| \le c\|v^{n-1}_{xx}\| \le c\|v^{n-1}\|_2 \le c\|u_0\|_2$ using (2.5) for $s = 2$. Since $A(t)$ is increasing, there clearly exists an upper bound for $\|u^1(t)\|_2$, $0 \le t \le T_0$. Let b_2 be such a bound.

Using (1.5), we obtain

$$\|\phi + u^1\|_\infty \le C\|\phi + u^1\|_1 \le C(\|u_0\|_1 + b_2) = d_0 \tag{2.8}$$

for $0 \le t \le T_0$ where we have used $\|u^1\|_2 \le b_2$. We then have $\|b_1(\phi + u^1)\|_\infty < d_1$ for some $d_1 > 0$, $0 \le t \le T_0$. Using (2.7) for $n = 2$ and Lemma 1.2 for $k = 1$,

$$\begin{aligned}\|u^2(t)\|_2 &\le cA(t)\{[M(t, d_0) + M_1(b_2, d_0)]\|\phi + u^1\|_1 + d_1\|\phi + u^1\|_2\} \\ &\le c_3 A(t) D_2(\|u_0\|_2 + b_2)\end{aligned} \tag{2.9}$$

for $0 \le t \le T_0$ where $D_2 = M_1(f, d_0) + M_1(b_2, d_0) + d_1$. If $t_0 \le T_0$ is chosen so that $A(t_0) \le b_2/(c_3 D_2(\|u_0\|_2 + b_2))$, then from (2.9), we see $\|u^2(t)\|_2 \le b_2$ for $0 \le t \le t_0$.

We now show by induction on n, $n \ge 3$, that $\|u^n(t)\|_2 \le b_2$ for $0 \le t \le t_0$. We have $\|\phi + u^{n-1}\|_\infty \le d_0$ for $0 \le t \le t_0$ since $\|u^{n-1}\|_1 \le \|u^{n-1}\|_2 \le b_2$. Therefore $\|b_1(\phi + u^{n-1})\|_\infty < d_1$ and from (2.7) we obtain $\|u^n(t)\|_2 \le cA(t)D_2(\|u_0\|_2 + b_2)$ in a fashion similar to (2.9). Hence, by the choice of t_0, $\|u^n(t)\|_2 \le b_2$ for $0 \le t \le t_0$.

We now prove the lemma for $k \ge 3$. We apply (2.6) to (2.31) for $w = u^n$, $a(u, t) = a_{n-1}(u, t)$ and use the triangle inequality to obtain

$$(2.10)\quad \|u^n(t)\|_k \le cA(t) \max_{0\le t\le t_0} \{\|f(v^{n-1})\|_{k-1} + \|b_2(v^{n-1})\|_{k-1} + \|b_1(v^{n-1})v_{xx}^{n-1}\|_{k-3}\}$$

for $0 \le t \le t_0$. Since $\|\phi + n^{n-1}\|_\infty \le d_0$ and therefore $\|b_1(\phi + n^{n-1})\|_\infty \le d_1$, we obtain from (2.10)

$$(2.11)\quad \begin{aligned}\|u^n(t)\|_k \le cA(t) \max_{0\le t\le t_0} \{&(M_{k-1}(f, d_0) + M_{k-1}(b, d_0))(1 + \|\phi + u^{n-1}\|_{k-2})^{k-2} \\ &+ d_1\|\phi + u^{n-1}\|_{k-1}\}\end{aligned}$$

for $0 \le t \le t_0$. But since $\|\phi + u^{n-1}\|_{k-2} \le \|\phi + u^{n-1}\|_{k-1} \le \|u_0\|_{k-1} + b_{k-1}$, it is clear from (2.11) that $\|u^n(t)\|_k \le c$ for $0 \le t \le t_0$ and the proof of the lemma is complete. ■

Lemma 2.4. *Assume $u_0 \in H^{s-1}$ if $s \ge 3$ or $u_0 \in H^s$ if $s = 2$. Then for some T^s and some ρ, $0 < \rho < 1$,*

$$(2.12)\qquad \sup_{0\le\tau\le T^s} \|u^{n+1}(\tau) - u^n(\tau)\|_s < \rho \sup_{0\le\tau\le T^s} \|u^n(\tau) - u^{n-1}(\tau)\|_s.$$

Proof. Subtracting (2.2) for n from (2.2) for $n + 1$ and using (2.4), we get

$$(u^{n+1} - u^n)_t + (u^{n+1} - u^n)_{xxx} = -(u^{n+1} - u^n)_{xxxx} + r(u^n, u^{n-1})$$

where $r(u^n, u^{n-1}) = a_n(x, t) - a_{n-1}(x, t)$. Using Lemma 2.2 for $w = u^{n+1} - u^n$, $a = r(u^n, u^{n-1})$, we obtain

$$(2.13)\qquad \|u^{n+1}(t) - u^n(t)\|_s \le cA(t) \sup_{0\le\tau\le t} \|r(u^n, u^{n-1})\|_{s-3}.$$

If we can then show $\|r(u^n, u^{n-1})\|_{s-3} \le c_1\|u^n(t) - u^{n-1}(t)\|_s$ for $0 \le t \le t_0$, then by choosing T^s so that $cA(T^s)c_1 = \rho < 1$, we will obtain (2.12).

Looking at the form of $r(u^n, u^{n-1})$, it clearly suffices to show:

$$(2.14)\qquad \|f(v) - f(w)\|_s \le c(T)\|v - w\|_s \quad \text{for } s \ge 0, 0 \le t \le T.$$

$$(2.15)\qquad \|f(v)v_{xx} - f(w)w_{xx}\|_{s-2} \le c(t)\|v - w\|_s \quad \text{for } s \ge 2,\ 0 \le t \le T.$$

The constant $c(t)$ will depend on $\|v(t)\|_s$ and $\|w(t)\|_s$, both of which are assumed bounded for $0 \le t \le T$. The proofs of (2.14) and (2.15) follow readily from the definition of the H^s norm using the fact that $f \in C^s$. ■

From Lemma 2.4, using the completeness of H^s, we may obtain a solution $u \in L^\infty(0, T; H^s)$ as a limit in H^s of the sequence $\{u^n\}$ along each t.

3. A priori estimates and global solutions to the perturbed problem. We extend the local solution of (1.4), (1.2) for each ϵ to a global solution in

Theorem 3.1. *Assume, in* (1.4), $f \in C^{m+1}$, g_1, g_2, $p \in C^m$ *satisfy* $lf(u) \geq 0$, $f(0) = 0$, $f'(u) \geq 0$, $p'(u) \leq 0$, $up(u) \leq 0$, $f(u)p(u) \leq 0$, *and* $g_1(u) + ug_1'(u) \geq ug_2(u)$, $g_1(u) \geq 0$, $g_2'(u) \leq 0$, $f'(u)g_1(u) + f(u)g_1'(u) \geq f(u)g_2(u)$, *and* $g_2'(u) - g_1''(u) \leq 0$. *Let* $u \in L^\infty(0, T^k; H^k)$ *for* $k = m + 5$ *for some* $T^k > 0$ *be a solution to* (1.4), (1.2). *Then we have* $u \in L^\infty(0, T; H^m)$ *for all* $T > 0$ *a solution to* (1.4), (1.2) *and*

$$\|u(t)\|_n \leq c(T; \|u_0\|_n) \quad \text{for } 0 \leq t \leq T,\ n = 0, \cdots, m, \tag{3.1}$$

$$\epsilon \int_0^t \|u(r)\|_{n+2}^2 \, dr \leq c(T; \|u_0\|_n) \quad \text{for } 0 \leq t \leq T,\ n = 0, 2, 3, \cdots, m, \tag{3.2}$$

where the constants are independent of ϵ.

We first have an important example in

Corollary 3.1. *Theorem* 3.1 *is true for* $f(u) = u^{2n+1}$, $g_1(u) = u^{2m}$, $g_2(u) = -u^{2r+1}$, $p(u) = -u^{2s+1}$ *for* n, m, r, s *nonnegative integers.*

Remark 3.1. The assumption $f(0) = 0$ is without loss of generality since $f(u)$ occurs in conservation form.

Remark 3.2. If we consider the perturbation $-(u_{xxxx} + f''(u)u_x^2)$ rather than $-(u_{xxxx} - f(u)_{xx})$, then under the additional hypotheses $uf''(u) \geq 0$ and $f(u)f''(u) \geq 0$, we may obtain the estimate (3.2) for $n = 1$. We did this in [2]. Note that Corollary 3.1 still applies under these additional hypotheses.

Remark 3.3. Notice that our hypotheses do not include $f(u) = u^2/2$, $p(u) = 0$ as in the K-dV itself. This is not surprising, as the cases $f(u) = u^2$, u^4 for $g_1(u) = 1$, $g_2(u) = 0$ in Tsutsumi and Mukasa (and elsewhere) where studied by special methods which do not generalize to $f(u) = u^n$ for $n \geq 5$. We may modify the proofs of Theorem 2.1 and 3.1 to obtain similar global existence of a solution to

$$u_t - uu_x + u_{xxx} = g_1(u)u_{xx} + g_2(u)(u_x)^2, \qquad u(x, 0) = u_0(x), \tag{3.3}$$

where g_1, g_2 are smooth and satisfy $g_1(u) + ug_1'(u) \geq ug_2(u)$, $g_1(u) \geq 0$, $g_2'(u) \leq 0$, and $2ug_1(u) + u^2g_1'(u) \geq u^2g_2'(u)$.

Proof of Theorem 3.1. In order to make a priori estimates on the L^2 norms of the rth order space derivatives, $D^r u$, we will consider

$$\tfrac{1}{2} d\|D^r u\|^2/dt = (D^r u, D_t D^r u) = (D^r u, D^r D_t u), \tag{3.4}$$

where the second equality of (3.4) is proved by noting that u is a limit in H^{r+5} strongly of solutions u^n to the linear equation (2.2). The identity (3.4) is useful, since $D_t u$ may be written as a function of u and its space derivatives by the differential equation (1.4).

We now turn to the proof of the estimates (3.1) and (3.2). The global existence of a solution follows readily from these. Fix $T > 0$. We will show $u(x, t) \in L^\infty(0, T; H^n)$ assuming $u(x, t) \in L^\infty(0, T; H^{n-1})$ for $n \geq 1$. We will find functional F, where possibly $F = 0$, so that

$$(3.5) \quad \tfrac{1}{2} d\|D^n u\|^2/dt + dF(u, Du, \cdots, D^n u)/dt \leq c(\|D^n u\|^2 + 1 - \epsilon\|D^{n+2}u\|^2)$$

for $0 \leq t \leq T$ holds, and either

$$(3.6a) \qquad |F(u, Du, \cdots, D^n u)| \leq c(\|D^n u\|^\alpha + 1), \quad 0 \leq \alpha < 2,$$

holds or

$$(3.6b) \qquad F(u) \geq 0 \quad \text{and} \quad |F(u_0, Du_0, \cdots, D^n u_0)| \leq c(\|u_0\|_n).$$

Integrating (3.5) and applying (3.6a) or (3.6b) and noticing $\|D^n u\|^\alpha \leq \tfrac{1}{2}\|D^n u\|^2 + c$, for c independent of each α, we may obtain (3.1) for n. We then obtain (3.2) for n from (3.5) using (3.1) for n.

We will prove separately $n = 0, 1, 2, 3$. The proof for $n \geq 4$ will be by an induction argument. Many of our special assumptions on f, g_1, g_2, and p are used in proving $n = 0, 1$. To prove $n = 2, 3$, carefully chosen functionals F must be used.

We first consider

$n = 0$.

$$(3.7) \quad \begin{aligned} \tfrac{1}{2} d\|u\|^2/dt = (u, u_t) &= (u, f(u)_x - u_{xxx} + g_1(u)u_{xx} + g_2(u)u_x^2 + p(u) \\ &\qquad - \epsilon(u_{xxxx} - f(u)_{xx})) \\ &= -(If(u)_x, 1) - (u_x^2, g_1(u) + ug_1'(u) - ug_2(u) + f'(u)) \\ &\quad + (u, p(u)) - \|u_{xx}\|^2. \end{aligned}$$

Since $-(If(u)_x, 1) = 0$ and by our hypotheses on g_1, g_2, f, and p, we see (3.5) follows for $F = 0$.

$n = 1$. In writing $(u_x, u_{xt}) = (u_x, u_{tx})$, we encounter the term $(u_x, f(u)_{xx})$ which cannot readily be estimated. Therefore we write, using integrations by parts,

$$(3.8) \quad \begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u_x\|^2 + \int If(u)\,dx\right) &= (u_x, f(u)_{xx}) - (g_1(u), u_{xx}^2) + \frac{1}{3}(g_2'(u), u_x^4) \\ &\quad + (f(u), f(u)_x) + (f(u), -u_{xxx}) \\ &\quad - (f'(u)g_1(u) + f(u)g_1'(u) - f(u)g_2(u), u_x^2) \\ &\quad - \epsilon\|u_{xxx} - f(u)_x\|^2. \end{aligned}$$

The first and fifth terms cancel as designed. Our hypotheses make the second, third, fourth, sixth and seventh terms nonpositive. Note that it is to write the right-hand side of (3.8) that we have used the perturbation $-\epsilon(u_{xxxx}-f(u)_{xx})$ rather than $-\epsilon u_{xxxx}$. With the perturbation $-\epsilon u_{xxxx}$, stronger hypotheses on $f(u)$ would be necessary, using our methods.

We then have

$$\frac{d}{dt}\left(\frac{1}{2}\|u_x\|^2+\int If(u)\,dx\right)\leq 0 \quad\text{and}\quad \int If(u)\,dx\geq 0 \quad\text{and}\quad \left|\int If(u(0))\,dx\right|\leq c(\|u_0\|_1).$$

To see the last inequality, we expand $If(u_0(x))$ in a Taylor series of second order about 0. From (3.9) and (3.6b) holding for $F(u)=\int If(u)\,dx$, we may derive (3.1) for $n=1$ as usual. However, since (3.9) does not contain the term $-\|u_{xxx}\|^2$ as (3.5) does for $n=1$, we may not readily derive (3.2) for $n=1$.

$n=2$. Using integrations by parts and the interchange of derivatives, we obtain

$$\begin{aligned}\tfrac{1}{2}\,d\|u_{xx}\|^2/dt &= 5(f''(u)u_x,\,u_{xx}^2)/2+(g_2''(u)u_x^4,\,u_{xx})+3(g_2'(u)u_x^2,\,u_{xx}^2)\\ &\quad-(2g_2(u)+g_1'(u),\,u_xu_{xx}u_{xxx})-(u_{xxx}^2,\,g_1(u))\\ &\quad-(u_x^5,\,f^{(4)}(u))/4+(u_{xx},\,p(u)_{xx}-\epsilon D^6u+\epsilon f(u)_{xxxx}).\end{aligned}\tag{3.10}$$

Since $\|u_x\|_\infty\leq c(\|u_{xx}\|^{1/2}+1)$ by Lemma 1.2, we may estimate the sum of the second and sixth terms of (3.10) and $(u_{xx},\,p(u)_{xx})$ from the ninth term of (3.10) by $c(\|u_{xx}\|^2+1)$. We also have $\epsilon(u_{xx},\,f(u)_{xxxx})=\epsilon(u_{xxxx},\,f(u)_{xx})\leq\epsilon\|u_{xxxx}\|\,\|f'(u)u_{xx}+f''(u)u_x^2\|\leq c\epsilon(\|u_{xxxx}\|^{3/2}+\|u_{xxxx}\|^{7/6}+1)\leq c\epsilon(\|u_{xxxx}\|^{3/2}+1)$ by using Lemma 2.1 on u_x for $p=0,\,1$ and $s=3$. Therefore, from (3.10) we derive

$$\begin{aligned}\tfrac{1}{2}\,d\|u_{xx}\|^2/dt &\leq 5(f''(u)u_x,\,u_{xx}^2)/2-(u_xu_{xx}u_{xxx},\,2g_2(u)+g_1'(u))\\ &\quad+3(g_2'(u)u_x^2,\,u_{xx}^2)+c(\|u_{xx}\|^2-\epsilon\|D^4u\|^2+1).\end{aligned}\tag{3.11}$$

To cancel the second term of (3.11) we consider, for $a(u)=I2g_2(u)+Ig_1'(u)$,

$$\frac{d}{dt}\int a(u)u_xu_{xx}\,dx=(a'(u)u_xu_{xx},\,u_t)-(a'(u)u_x^2,\,u_{tx})\tag{3.12}$$

where we have used the interchange of derivatives. Then substituting u_t from (1.4), we find that the terms of the right-hand side of (3.12) may be estimated by $-3(g_1'(u)+2g_2(u),\,u_xu_{xx}u_{xxx})+3(g_1''(u)+2g_2'(u),\,u_x^2u_{xx}^2)+3(a'(u)g_1(u)u_x,\,u_{xx}^2)+c\epsilon(\|u_{xxxx}\|^{3/2}+1)+c(\|u_{xx}\|^2+1)$. The estimates are derived using Lemma 1.2 as before.

Therefore,

$$(3.13)\quad \frac{d}{dt}\left(\frac{1}{2}\|u_{xx}\|^2 - \frac{1}{3}\int a(u)u_x u_{xx}\,dx\right) \le (5f''(u)/2 - a'(u)g_1(u),\ u_x u_{xx}^2) + c(\|u_{xx}\|^2 - \epsilon\|u_{xxxx}\|^2 + 1).$$

We therefore consider, for

$$b(u) = I\left(\frac{5}{2}f''(u) - a'(u)g_1(u)\right)$$

$d\int b(u)u_x^2\,dx/dt$ and find that it equals $-3(b'(u),\ u_x u_{xx}^2)$ plus terms which may be estimated using Lemma 1.2 as before. Therefore, from (3.13) we obtain (3.5) where

$$F(u) = -\frac{1}{3}\int a(u)u_x u_{xx}\,dx + \frac{1}{3}\int b(u)u_x^2\,dx.$$

Then since

$$\left|-\frac{1}{3}\int a(u)u_x u_{xx}\,dx\right| \le \|a(u)\|_\infty \|u_x\|\|u_{xx}\| \le c\|u_{xx}\|$$

and

$$\left|\frac{1}{3}\int b(u)u_x^2\,dx\right| \le \|b(u)\|_\infty \|u_x\|^2 \le c,$$

we have (3.6a) holding for $\alpha = 1$.

$n = 3$. After expanding u_{txxx} by writing u_t from (1.4) and differentiating three times, and after performing many integrations by parts, we obtain

$$(3.14)\quad \tfrac{1}{2}\,d\|u_{xxx}\|^2/dt \le (3g_1'(u) + 5g_2(u),\ u_{xx}u_{xxx}^2) + c(\|u_{xxx}\|^2 - \epsilon\|D^5u\|^2 + 1).$$

We have used Lemma 2.1 on the function u_{xx} for $p = 0$, $s = 1$ and on u_x for $p = 1$, $s = 2$ in estimating the various terms of (3.14). Let $d(u) = I(3g_1'(u) + 5g_2(u))$ and consider $F(u) = \int d(u)u_{xx}u_{xxx}\,dx$. Then

$$(3.15)\qquad dF(u)/dt = (d'(u)u_{xx}u_{xxx},\ u_t) - (d'(u)u_x u_{xx},\ u_{txx}).$$

Substituting u_t from (1.4) and estimating the resulting terms of (3.14) as before using Lemma 2.1, we find that $dF(u)/dt$ equals $\tfrac{1}{2}(d(u),\ u_{xx}u_{xxx}^2)$ plus terms which can be estimated by $c(\|u_{xxx}\|^2 + 1)$ or by $c\epsilon(\|D^5u\|^\beta + 1)$ for $\beta \le 11/6$. Therefore,

$$\frac{d}{dt}\left(\frac{1}{2}\|u_{xxx}\|^2 - 2\int d(u)u_{xx}u_{xxx}\,dx\right) \le c(\|u_{xxx}\|^2 - \epsilon\|D^5u\|^2 + 1).$$

And since $|-2\int d(u)u_{xx}u_{xxx}\,dx| \le c\|u_{xxx}\|$, we have (3.5) and (3.6a).

$n \ge 4$. For $n \ge 4$, we have

$$\tfrac{1}{2}\, d\|D^n u\|^2/dt = (D^n u,\, D^n u_t) \le (-g_1(u),\, (D^{n+1}u)^2) + c(\|D^n u\|^2 - \epsilon\|D^{n+2}u\|^2 + 1)$$

for $0 \le t \le T$ where we have assumed $\|u(t)\|_{n-1} \le c$ for $0 \le t \le T$. This is readily seen by expanding $D^n u_t$ from (1.4) and using Lemma 1.2 repeatedly on the resulting terms.

This completes the proof of Theorem 3.1. For full details, see [2].

Using the methods of the proof of Theorem 3.1, we may obtain similar a priori estimates for (1.1), (1.2) in the following.

Theorem 3.2. *Under the same hypotheses as Theorem* 3.1 *but not including* $f'(u) \ge 0$, *we have estimates* (3.1) *and* (3.2).

4. Global existence and uniqueness for the unperturbed problem.

Theorem 4.1. *If* $u_0 \in H^s$ *and if* $f \in C^{s+3}$, $g_1, g_2, p \in C^{s+2}$ *and they satisfy the hypotheses of Theorem* 3.1, *then the problem* (1.1), (1.2) *has a solution* $u(x, t) \in L^\infty(0, T; H^s)$ *for each* $T > 0$ *if* $s \ge 6$. *If* $s \ge 7$, *then* u *is a genuine solution.*

Proof. First choose $\{u_{0,n}\}_{n=1}^\infty \subseteq C_0^\infty$ so that $u_{0,n} \to u_0$ in H^s as $n \to \infty$. Let u_n be the solution to (1.4), (1.2) for data $u_{0,n}$ and $\epsilon = 1/n$. Notice that $\epsilon \to 0$ as $n \to \infty$. We then have from Theorem 3.1

$$\|u_n(t)\|_{s+3} \le c(\|u_{0,n}\|_{s+3}, T) \le c(\|u_0\|_{s+3}, T), \tag{4.1}$$

$$\epsilon \int_0^t \|u_n(\tau)\|_{s+5}^2\, d\tau \le c(\|u_{0,n}\|_{s+3}, T) \le c(\|u_0\|_{s+3}, T), \tag{4.2}$$

for $0 \le t \le T$, $n = 1, 2, \cdots$.

Choose a set of lines $\{t_m\}_{m=1}^\infty$ which are dense in $[0, \infty)$. Let $H^s(\Omega)$ be the space of functions u in $L^2(\Omega)$ which have L^2-derivative of order up to s with the usual norm. Then by induction and the reflexivity of H^s, choose a subsequence of $\{u_n\}$, call it $\{u_n\}$, so that for each m and each positive integer M,

$$u_n(x, t_m) \to u(x, t_m) \quad \text{weakly in } H^s \text{ and} \tag{4.3}$$

$$\|u_n(t_m)\|_{H^s(-M,M)} \quad \text{converges in } R^1. \tag{4.4}$$

Fix t_m; by Rellich's theorem, for each compact subset A of R^1 we have a subsequence $\{u_{n_j}\} \subseteq \{u_n\}$ so that $u_{n_j}(x, t_m) \to u(x, t_m)$ strongly in $H^{s-1}(A)$. By (4.4), we have $u_n(x, t_m) \to u(x, t_m)$ strongly in $H^{s-1}(-M, M)$.

Following Oleinik [8, p. 122], we define a limit function $u(x, t)$ for all $t > 0$. Consider, for $(-M, M) = A$,

$$\|u_n(t)-u_l(t)\|_{H^{s-3}(A)} \le \|u_n(t)-u_n(t_m)\|_{H^{s-3}(A)} + \|u_n(t_m)-u_l(t_m)\|_{H^{s-3}(A)} \tag{4.5}$$
$$+ \|u_l(t)-u_l(t_m)\|_{H^{s-3}(A)}.$$

For $|t-t_m|<1$ and $T=t+1$,

$$\|u_n(t)-u_n(t_m)\|^2_{H^{s-3}(A)} \le |t-t_m| \int_0^t \|(u_n)_t(\tau)\|^2_{H^{s-3}(A)}\, d\tau.$$

Substituting $(u_n)_t$ from the equation (1.4) and using (4.1) for s and (4.2) for $s-1$, we see

$$\|u_n(t)-u_n(t_m)\|_{H^s(A)} \le c(t)|t-t_m|^{1/2}, \tag{4.6}$$

where $c(t)$ is independent of n and A. Using (4.6) we may choose t_m so that the first and third terms of (4.5) are small. We may then make the second term small by choosing n, l sufficiently large because u_n is strongly convergent in $H^s(A)$ along t_m.

For $s \ge 6$, we have

$$\|(u_n)_x\| \le c, \quad \|(u_n)_{xx}\| \le c, \quad \|(u_n)_t\| \le c, \quad \|(u_n)_{tt}\| \le c.$$

Therefore, choose a subsequence of $\{u_n\}$, which we again call $\{u_n\}$, so that $u_n \to v$ in $L^\infty(0, T; H^s)$ weak star and $u_n \to v$ in $H^1(Q)$ for Q a compact subset of $t \ge 0$. Then since $u_n \to u$ in $H^3(A)$ along each t, for A a compact subset of R^1, then $u = v$. Hence, $u \in L^\infty(0, T; H^s)$.

We now wish to show $u(x, t)$ is a genuine solution if $s \ge 7$. We have $\lim_{n\to\infty} D^p u_n = D^p u$ for $p = 0, 1, 2, 3$. Since $\|D^p u\|_\infty < c\|u\|_5$ for $p = 0, 1, 4$, then $-\epsilon_n((u_n)_{xxxx} - f(u_n)_{xx}) \to 0$ as $n \to \infty$. To show $\lim_n (u_n)_t = u_t$, consider

$$u_n(x, t_1) - u_n(x, t_2) = \int_{t_1}^{t_2} (u_n)_t(x, \tau)\, d\tau.$$

We have $u_n(x, t_i) \to u(x, t_i)$ for $i = 1, 2$, and we have

$$\|(u_n)_t(x, t_1) - (u_n)_t(x, t_2)\|^2_\infty \le c\|(u_n)_t(x, t_1) - (u_n)_t(x, t_2)\|^2_1 \tag{4.7}$$
$$\le c|t_1 - t_2| \int_0^T \|(u_n)_{tt}(\tau)\|^2_1\, d\tau \le c|t_1 - t_2|.$$

Expanding u_{tt} using the differential equation and then using (3.1) and (3.2) for $m = 7$, we obtain the last line of (4.7). Therefore the $(u_n)_t$ are (equi)uniformly continuous in time. Examining

$$\|(u_n)_t(x, t) - u_t(x, t)\|^2_\infty \le c\|(u_n)_t(x, t) - u_t(x, t)\|^2_1 \quad \text{for } x \in A \tag{4.8}$$

we see that $(u_n)_t \to u_t$ uniformly on any compact set assuming $s \geq 7$. This completes the proof of the theorem. ■

Remark 4.1. We have obtained the convergence of a subsequence of the original $\{u_n\}$. Since we will show the limit function is unique, the entire sequence $\{u_n\}$ will converge to the solution of (1.1), (1.2).

Remark 4.2. We have passed to the limit in a fashion similar to the methods of Temam [10] or Tsutsumi et al. [11]. These methods lead to convergence in $L^2(0, T; H^{s-1}(Q))$, whereas ours also gives convergence in $L^\infty(0, T; H^{s-3}(Q))$ for Q a compact set of $t \geq 0$.

Because the hypotheses for uniqueness are so much weaker than for existence, we state the following

Theorem 4.2. *If* $u \in L^\infty(0, T; H^3)$ *is a solution to* (1.1), (1.2) *or* (1.4), (1.2) *and if* $g_1(u) \geq 0$, *then it is unique.*

Proof. Let u, v be solutions to either problem (1.1), (1.2) or (1.4), (1.2) with data u_0. Writing, for $w = u - v$, from (1.1) or (1.4)

$$\tfrac{1}{2} d\|w\|^2/dt = (w, w_t) \leq c(T)\|w\|^2 \tag{4.9}$$

where we understand $c(T)$ depends on $\|u\|_3$ for $0 \leq t \leq T$. The proof is by Lemmas 1.1 and 1.2 as in §3. From (4.9) we obtain $w(x, t) = 0$ a.e. in x for $0 \leq t \leq T$. Then since $w \in H^3$ for each t, $0 \leq t \leq T$, $w \in C^2$ as a function of x for each t. Hence, $w \equiv 0$. Since T was arbitrary, the proof of the theorem is complete.

5. **An analogous development for periodic initial value problems.** In order to state and prove theorems corresponding to those of §§1–4, we must use appropriate Sobolev spaces. Let $\mathring{C}^\infty$ be the space of functions in C^∞ of period 2π; let $L^2(0, 2\pi) = \mathring{L}^2$. Let $\mathring{H}^s$ be the completion of $\mathring{C}^\infty$ with respect to the norm $\| \|_{s, A}$, $A = (0, 2\pi)$. Thereafter, we abbreviate the norm in $\mathring{H}^s$ by $\| \|_s$ and the norm in $\mathring{L}^2$ by $\| \|$ the space $L^\infty(0, T; \mathring{H}^s)$ is defined analogously to $L^\infty(0, T; H^s)$.

We have the following local existence theorem:

Theorem 5.1. *Let* $s \geq 3$ *and let* $u_0 \in \mathring{H}^{s-1}$, $f \in C^{s-1}$, $g_1, g_2, p_0 \in C^{s-2}$. *Then the problem* (1.4), (1.2), (1.3) *has a solution* $u(x, t) \in L^\infty(0, T; H^s)$ *for some* $T = T(s)$.

Proof. We follow the development of the proof of Theorem 2.1. Let ψ be a solution to the problem

$$\begin{aligned} &\psi_t + \psi_{xxx} = -\epsilon\psi_{xxxx}, \\ &\psi(x, 0) = n_0(x), \quad \psi(x + 2\pi, t) = \psi(x, t) \quad \text{for } t \geq 0 \end{aligned} \tag{5.1}$$

and let $\{u^n\}$ be defined as we did in (2.2), but where $v^{n-1} = \psi + u^{n-1}$ and we require of each u^n that $u^n(x + 2\pi, t) = u^n(x, t)$.

We now have the periodic analogue of Lemma 2.1.

Lemma 5.1. *Let $s \geq 1$ and let $u_0 \in \overset{\circ}{H}{}^s$. Then $\psi(x, t) \in L^\infty(0, T; \overset{\circ}{H}{}^s)$ for all $T > 0$ and*

$$\|\psi(t)\|_k \leq \|u_0\|_k \quad \textit{for } k = 1, \cdots, s \textit{ and } t \geq 0. \tag{5.2}$$

Proof. Let

$$E(x, t) = \sum_{n=-\infty}^{\infty} \exp(inx - \epsilon n^4 t + in^3 t)$$

be a fundamental solution to (5.1). Then we have a distribution solution $\psi(x, t) = E(x, t) * u_0(x)$ where $*$ in this section denotes convolution in $[0, 2\pi]$. Since $u_0 \in \overset{\circ}{H}{}^s$ for $s \geq 1$, it has the L^2 representation $u_0(x) = \Sigma\, a_m e^{imx}$, so we have

$$\|D^k\psi(t)\|^2 = \sum \exp(-2\epsilon n^4 t) n^{2k} a_n^2 \leq \|D^k u_0\|^2.$$

We consider the periodic analogue of (2.3), for $a(x, t) = a(x + 2\pi, t)$ for all $t \geq 0$

$$\begin{aligned} &v_t + v_{xxx} = -\epsilon D^4 v + a(x, t), \\ &v(x, 0) = 0, \qquad v(x + 2\pi, t) = v(x, t) \quad \text{for } t \geq 0. \end{aligned} \tag{5.3}$$

For (5.3) we have

Lemma 5.2. *Let $s \geq 3$, and let $a(x, t) \in L^\infty(0, T; \overset{\circ}{H}{}^{s-3})$, then $v(x, t) \in L^\infty(0, T; \overset{\circ}{H}{}^3)$ and satisfies*

$$\|v(t)\|_k \leq c(\epsilon) A(t) \sup_{0 \leq \tau \leq t} \|a(\tau)\|_{k-3} \quad \textit{for } k = 3, 4, \cdots, s \tag{5.4}$$

where $A(t)$ is as defined in Lemma 2.2.

Proof. Letting $E(x, t)$ again denote a fundamental solution to (5.1), then $v(x, t) = \int_0^t E(x, t - \tau) * a(x, \tau)\, d\tau$ is a distribution solution to (5.3). Then by Parseval's identity, we get

$$\begin{aligned} \|D^k v(t)\| &\leq \int_0^t \|E(x, t - \tau)^* D^k a(x, \tau)\|\, d\tau \\ &\leq \int_0^t \sum_{n=-\infty}^{\infty} ((\epsilon(t - \tau))^{-3/2} n^6 (\epsilon(t - \tau))^{3/2} \exp(-2\epsilon n^4(t - \tau)) n^{2k-6} a_n^2(\tau))^2\, d\tau. \end{aligned}$$

Then letting $n^2\epsilon(t-\tau)^{1/2} = y$, we have $y^3\exp(-2y^2) < c_1^2$ for all n. Hence

$$\|D^k v(t)\| \le c_1 \int_0^t (\epsilon(t-\tau))^{-3/4}\, d\tau \sup_{0\le\tau\le t} \left(\sum_{n=-\infty}^{\infty} n^{2(k-3)} a_n^2(\tau) \right)^{1/2}$$

$$\le c(\epsilon)t^{1/4} \sup_{0\le\tau\le t} \|a(\tau)\|_{k-3} \le c(\epsilon)A(t) \sup_{0\le\tau\le t} \|a(\tau)\|_{k-3}.$$

We now have the machinery to prove the following:

Lemma 5.3. *Let* $s \ge 3$ *and let* $u_0 \in \mathring{H}^{s-1}$. *For* $\{u^n\}$ *we have*

(5.5) $\|u^n(t)\|_k \le c(k, t)$ *for* $0 \le t \le t_0$ *for some* t_0 *and for* $k = 3, 4, \cdots, s$.

Lemma 5.4. *Let* $s \ge 3$ *and let* $u_0 \in \mathring{H}^{s-1}$. *For some* T^s, *and some* ρ, $0 < \rho < 1$ *we have*

$$\sup_{0\le\tau\le T^s} \|u^{n+1}(\tau) - u^n(\tau)\|_s < \rho \sup_{0\le\tau\le T^s} \|u^n(\tau) - u^{n-1}(\tau)\|_s.$$

Remarks on the proofs. We prove (5.5) by induction on k, starting with $k = 3$. Formally Lemmas 5.3 and 5.4 are the same as 2.1 and 2.2. Since we may identify $\mathring{H}^s$ with a subspace of H^s, Lemmas 1.1 and 1.2 obtain, where $\|\ \|_s$ means the norm in $\mathring{H}^s$. With these remarks, replacing ϕ with ψ and using $\|\ \|_s$ to mean the $\mathring{H}^s$ norm rather than the H^s norm, the proof of Lemma 5.3 is closely analogous to that of Lemma 2.3 for $k \ge 3$.

Lemma 5.4 is proved much as we did Lemma 2.4, using Lemmas 5.3 and 5.4 in place of 2.1 and 2.2.

We complete the proof of Theorem 5.1 by using the completeness of $\mathring{H}^s$.

We have global existence for the perturbed problem through a priori estimates in the following

Theorem 5.2. *For the problems* (1.4), (1.2), (1.3) $\{$(1.1), (1.2), (1.3)$\}$, *let the hypotheses of Theorem* 3.1 $\{$*Theorem* 3.2$\}$ *on* f, g_1, g_2, *and* p *hold, then if a local solution* $u \in L^\infty(0, T; \mathring{H}^{m+5})$ *exists for some* $T > 0$, *we have* (3.1) *and* (3.2) *where* $\|\ \|_s$ *denotes the* $\mathring{H}^s$ *norm.*

Proof. The assumption $u \in L^\infty(0, T; \mathring{H}^{m+5})$ guarantees that $D^p u(x, t) = D^p u(x + 2\pi, t)$ for $0 \le p \le m + 4$. This insures the cancellation of the boundary terms occurring in the integrations by parts in the computations used to prove Theorem 3.1. Otherwise, we may use the proof of Theorem 3.1 to prove Theorem 5.2.

By a procedure very similar to the one used in Theorem 4.1 we may prove

Theorem 5.3. *For problems* (1.1), (1.2), (1.3), *if* f, g_1, g_2, *and* p *satisfy the hypotheses of Theorem* 3.1 *and if* $f \in C^{s+3}$, $g_1, g_2, p \in C^{s+2}$ *and* $u_0 \in \overset{\circ}{H}{}^s$, *then there exists a solution* $u(x, t) \in L^\infty(0, T; \overset{\circ}{H}{}^s)$ *for each* $T > 0$ *if* $s \geq 6$. *It is a genuine solution for* $s \geq 7$.

And following Theorem 4.2, we may show

Theorem 5.4. *If* $u(x, t) \in L^\infty(0, T; \overset{\circ}{H}{}^3)$ *is a solution to* (1.1), (1.2), (1.3) *or* (1.4), (1.2), (1.3) *and* $g_1(u) \geq 0$, *then* u *is unique.*

We will now show that the solution to the perturbed problems (1.4), (1.2), (1.3) may be obtained as a limit of solutions to a differential-difference scheme, using the methods and notation of Sjoberg [9].

For each integer M, let $h = M^{-1}$ be the mesh width (in space), where D_+ and D_- are defined as usual. The approximating functions $u_M = u_M(x_r, t)$ for $r = 1, \cdots, M$ are defined by the differential-difference scheme

$$
\begin{aligned}
du_M/dt = D_+f(u_M) - D_+^2D_-u_M + g_1(u_M)D_+D_-u_M& \\
+ g_2(u_M)((D_+u_M)^2 + (D_+u_M)(D_-u_M) + (D_-u_M)^2) - \epsilon(D_+^2D_-^2u_M - D_+D_-f(u_M))&, \\
u_M(x_r, 0) = u_0(x_r) \quad \text{for } r = 1, \cdots, M&, \\
u_M(x_r, t) = u_M(x_{r+M}, t) \quad \text{for } r = 1, \cdots, M \text{ and } t \geq 0&.
\end{aligned}
$$

In the sequel, we use the notation $(\, , \,) = (\, , \,)_u$ and $\| \; \| = \| \; \|_u$ to indicate the standard L^2 inner product and norm for periodic gridfunctions with mesh width h. Also let $\|u\|_\infty = \max_{i=1}^M |u(x_i)|$. In the proofs we abbreviate $u_M = u$.

We prove the following a priori estimates:

Lemma 5.5. *The solution* u_M *satisfies the following estimates for* $0 \leq t \leq T$ *independent of* h:

$$\|u_M\|_h \leq c, \tag{5.6}$$

$$\|D_+u_M\|_h \leq c, \tag{5.7}$$

$$\|D_+^mD_-^mu_M\|_h \leq c \quad \textit{for } m \geq 1. \tag{5.8}$$

Proof. To prove (5.6), we consider

$$
\begin{aligned}
\tfrac{1}{2}d\|u\|^2/dt = (u, u_t) = (u, D_+f(u) - D_+^2D_-u + g_1(u)D_+D_-u& \\
+ g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2) - \epsilon(D_+^2D_-^2u - D_+D_-f(u)))&.
\end{aligned} \tag{5.9}
$$

The hypotheses $f'(u) \geq 0$, $(ug_1(u))' \geq 0$, and $ug_2(u) \leq 0$ make the first and sixth, third, and fourth terms of (5.9) nonpositive. The second and fifth terms are evidently nonpositive. This proves (5.6).

Using our methods in Theorem 3.1 for $n = 1$, we see that

$$d(If(u_M), 1)/dt + \tfrac{1}{2}\|D_+u_M\|^2 = (f(u) - D_+D_-u,\ du/dt)$$

$$(5.10) \qquad = -(\epsilon + b/2)\|D_+f(u) - D_+^2D_-u\|^2 + (f(u) - D_+D_-u,\ g_1(u)D_+D_-u)$$

$$+ (f(u) - D_+D_-u,\ g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2)).$$

The second term of (5.10) is nonpositive if we assume $(f(u)g_1(u))' \geq 0$ and $g_1(u) \geq 0$, as we have. If we assume $f(u)g_2(u) \leq 0$ and $g_2'(u) \leq 0$, the third term of (5.10) will be nonpositive. To see that, consider that

$$-(D_+D_-u,\ g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2)) = -(g_2(u),\ D_-(D_+u)^3)$$
$$= ((D_+u)^3,\ D_+(g_2(u))) = ((D_+u)^4,\ g_2'(\xi_M))$$

where ξ_M is chosen between $u(x_r)$ and $u(x_{r+1})$ by the mean value theorem applied to $g_2(u)$.

From (5.10), we therefore obtain

$$(5.11) \qquad (If(u), 1) + \tfrac{1}{2}\|D_+u\|^2 \leq (If(u_0), 1) + \tfrac{1}{2}\|D_+u_0\|^2.$$

Assuming $If(u) \geq 0$, we obtain (5.7), since $|(If(u_0), 1)| \leq \max_x |f'(u_0(x))|\ \|u_0\|^2$.

We now prove (5.8) by induction on m for $m = 1, 2, \cdots$. For each m, we show

$$\tfrac{1}{2} d\|D_+^mD_-^mu\|^2/dt = (D_+^mD_-^m,\ D_+^mD_-^m du/dt) \leq -\epsilon\|D_+^{m+1}D_-^{m+1}u\|^2/4 + C_1,$$

where C_1 depends on ϵ, for $0 \leq t \leq T$. Then by Gronwald's inequality, the proof will be complete.

We first consider

$$(D_+D_-u,\ D_+D_-du/dt) = (D_+^2D_-u,\ D_+(f(u)) + g_1(u)D_+D_-u$$
$$(5.12) \qquad + g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2)$$
$$- \epsilon(D_+^2D_-^2u - D_+D_-(f(u)))).$$

To estimate (5.12), we will make use of the following

Lemma 5.6 (Kreiss [3]). *Let σ, τ be integers, $0 \leq \tau < \sigma$, $\sigma < M/2 - 1$. Then to every $\epsilon > 0$, there exists a $c(\epsilon)$ independent of u and h so that*

$$\max_{r=1}^{N} |D_+^\tau u(x_r)| \leq \epsilon\|D_+^\sigma u\|^2 + c(\epsilon)\|u\|_h^2.$$

This lemma is true for a periodic gridfunction $u(x_r)$, $r = 1, \cdots, M$; the operators D_+^τ and D_+^σ may be replaced by $D_+^{\tau_1}D_-^{\tau_2}$, $\tau_1 + \tau_2 = \tau$, or $D_+^{\sigma_1}D_-^{\sigma_2}$, $\sigma_1 + \sigma_2 = \sigma$, respectively.

Using Lemma 5.6, we have, since $\|u\|_\infty \leq c$,

$$(D_+^2D_-^2u,\ D_+f(u)) \leq c\|D_+^2D_-^2u\|\|D_+u\| \leq c_1\|D_+^2D_-^2u\|(\eta_1\|D_+^2D_-^2u\| + c(\eta_1)),$$

$$(D_+^2D_-^2u,\ g_1(u)D_+D_-u) \leq c_2\|D_+^2D_-^2u\|(\eta_2\|D_+^2D_-^2u\| + c(\eta_2)),$$

$$\begin{aligned}(D_+^2D_-^2u,\ g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2)) &\leq c\|D_+^2D_-^2u\|\|D_+u\|(3\|D_+u\|_\infty)\\ &\leq c_3\|D_+^2D_-^2u\|(\eta_3\|D_+^2D_-^2u\| + c(\eta_3)),\end{aligned}$$

$$\epsilon(D_+^2D_-^2u,\ D_+D_-f(u)) \leq \epsilon c_4\|D_+^2D_-^2u\|(\eta_4\|D_+^2D_-^2u\| + c(\eta_4)).$$

We then choose η_i, $i = 1, \cdots, 4$, in the above so that $c_i\eta_i < \epsilon/8$ so that from (5.12), we obtain $d\|D_+D_-u\|^2/dt \leq -\epsilon\|D_+D_-u\|^2/4 + c$.

For $m > 1$, the proof is by induction. We have, by Lemma 5.6, $\max_x |D_+^{r_1}D_-^{r_2}u| \leq c$ for $r_1 + r_2 \leq 2m - 3$. To prove (5.8) we consider

$$\begin{aligned}(D_+^mD_-^mu,\ D_+^mD_-^mu/dt) = (D_+^{m+1}D_-^{m+1}u,\ &D_+^mD_-^{m-1}(f(u)) + D_+^{m-1}D_-^{m-1}(g_1(u)D_+D_-u)\\ &+ D_+^{m-1}D_-^{m-1}(g_2(u)((D_+u)^2 + (D_+u)(D_-u) + (D_-u)^2))\\ &- (\epsilon + b/2)\|D_+^{m+1}D_-^{m+1}u\|^2 + \epsilon D_+^mD_-^m(f(u))).\end{aligned} \tag{5.13}$$

To estimate the second term of (5.13), we write

$$\begin{aligned}&(D_+^{m+1}D_-^{m+1}u,\ D_+^{m-1}D_-^{m-1}(g_1(u)(D_+D_-u))\\ &\qquad\leq c\|D_+^{m+1}D_-^{m+1}u\| \sum_{k=0}^{2m-2} \|D_+^{2m-2-k}g_1(u)\|\|D_+^{k+2}u\|\end{aligned}$$

using Leibnitz's formula for finite differences. We then estimate $\|D_+^rg_1(u)\| \leq \|D_+^rg_1(u)\|_\infty \leq c\sum \|D_+^{r_1}u\|_\infty \cdots \|D_+^{r_s}u\|_\infty$, where the sum is taken over all combinations $r_1, \cdots, r_s$ so that $r_1 + r_2 + \cdots + r_s = r$, and obtain:

$$\begin{aligned}&(D_+^{m+1}D_-^{m+1}u,\ D_+^{m-1}D_-^{m-1}(g_1(u)D_+D_-u))\\ &\qquad\leq c\|D_+^{m+1}D_-^{m+1}u\| \sum \|D_+^{r_1}u\|_\infty \cdots \|D_+^{r_s}u\|_\infty\|D_+^{k+2}u\|,\end{aligned} \tag{5.14}$$

where the sum in (5.14) is taken over all combinations $r_1, \cdots, r_s$ so that $r_1 + \cdots + r_s = 2n - 2 - k$, $k = 0, \cdots, 2n - 2$. In the sum in the right-hand side of (5.14), at most one factor from each term will have $r_j \geq 2n - 2$ or $k + 2 \geq 2n - 1$. This factor may be estimated by $\eta\|D_+^{m+1}D^{m+1}u\| + c(\eta)$. The other factors are bounded by constants, by induction. Therefore, for the right-hand side of (5.14) we obtain the estimate $\eta c\|D_+^{m+1}D_-^{m+1}u\|^2 + c\|D_+^{m+1}D_-^{m+1}u\|$.

By similar methods, we may estimate the other terms of (5.13) and bound the entire right-hand side by $-\epsilon\|D_+^{m+1}D_-^{m+1}u\|^2/2 + c_1\|D_+^{m+1}D_-^{m+1}u\| \leq -\epsilon\|D_+^{m+1}D_-^{m+1}u\|^2/4 + c_2$, which is what we set out to prove.

Using the a priori estimates of Lemma 5.5, we may readily obtain a solution to the perturbed problems (1.4), (1.2), (1.3) in

Theorem 5.5. *If* $u_0 \in \overset{\circ}{H}{}^{2k}$, $k \geq 4$, *then for each* $T > 0$, *there exists a solution* $u \in L^\infty(0, T; \overset{\circ}{H}{}^{2k})$.

Proof. Using Sjoberg's method of discrete Fourier series [9], we form, for $M = 2n + 1$, $\phi_n(x, t) = \sum_{\omega=-n}^{n} a_n(\omega, t)e^{2\pi i\omega x}$, $a_n(\omega, t) = (e^{2\pi i\delta x}, u_M(x, t))_h$. Then by our estimate (5.8) for $m = 4$, we find the functions ϕ_n satisfy the same a priori estimates, where we replace $D_+^m D_-^m$ by $\partial^{2m}/\partial x^{2m}$ and $\| \ \|_h$ by $\| \ \|$, the $L^2(0, 1)$ norm.

From this, we see $\partial^4\phi_n/\partial x^4$ are uniformly bounded and equicontinuous in any $[0, 1] \times [0, T]$. Using the Arzela-Ascoli selection theorem, we obtain a subsequence converging to the solution.

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